Cauchy, Cluster Points and Limits Infimum/Supremum

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Question 1 - Cauchy and Convergence

A sequence $(x_n) \in \mathbb{R}^{\mathbb{N}}$ is called *Cauchy* if and only if for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $n, m \geq N$ implies $|x_n - x_m| \leq \varepsilon$. Prove that $(x_n) \in \mathbb{R}^{\mathbb{N}}$ is Cauchy if and only if (x_n) converges to some point $a \in \mathbb{R}$.

Question 2 - Cluster Points

(Wade 2.4.7)

a) Let E be a subset of $\mathbb{R}$. A point $a \in \mathbb{R}$ is called a *cluster point* of E if $E \cap (a - r, a + r)$ contains infinitely-many points for every $r > 0$. Prove that a is a cluster point of E if and only if for each $r > 0$, $E \cap (a - r, a + r) \setminus \{a\}$ is nonempty.

b) Prove that every bounded infinite subset of $\mathbb{R}$ has at least one cluster point.

Question 3 - Limits Infumum/Supremum

The *limit supremum* of a real sequence $(x_n) \in \mathbb{R}^{\mathbb{N}}$ is the extended real number $\limsup_{n\to\infty} x_n := \lim_{n\to\infty} (\sup_{k\geq n} x_k)$ and the *limit infimum* of (x_n) is the extended real number $\liminf_{n\to\infty} := \lim_{n\to\infty} (\inf_{k\geq n} x_k)$. Suppose that (x_n) is a real sequence. Prove that

$$-\limsup_{x\to\infty} x_n = \liminf_{x\to\infty}(-x_n)$$

and

$$-\liminf_{x\to\infty} x_n = \liminf_{x\to\infty}(-x_n)$$