9. Similarity, Diagonalization and Normal Forms

We begin our discussion with a definition of similarity of matrices.

Definition 1 (Similarity)

We say that two matrices A and B are similar (denoted $A \sim B$) if there exists an invertible matrix P such that $A = P^{-1}BP$.

Now that we are equipped with the concept of matrix similarity, we can discuss diagonalizability.

Definition 2 (Diagonalizability)

We say that a matrix A is diagonalizable if there exists a diagonal matrix Λ such that $A \sim \Lambda$.

Theorem 1 (Criterion for Diagonalizability)

Let $A : V \to V$ be a linear transformation. The following are equivalent:

- A is diagonalizable.
- There exists a basis of eigenvectors of A.
- The geometric multiplicities and algebraic multiplicities are equal for all eigenvalues λ.

We shall now discuss matrices with complex eigenvalues.

Definition 3 (Complexification)

Let $T \in \mathcal{L}(\mathbb{R}^n)$. We define the complexification of T as $T_\mathbb{C} \in \mathcal{L}(\mathbb{C}^n)$ such that $T_\mathbb{C} = T$.

Definition 4 (Semisimple)

We say that a matrix $A \in \mathbb{M}_{n\times n}$ is semisimple if its complexification is diagonalizable.

Theorem 2

If $\lambda \in \mathbb{C}$ is an eigenvalue of A, then $\overline{\lambda}$ is also an eigenvalue.

Proof.
Assume that $\lambda = a + bi$ is an eigenvalue of A. Then there exists a nonzero eigenvector $\xi \in \mathbb{C}^n$ such that $A_\mathbb{C}\xi = \lambda\xi$. Taking the complex conjugate of both sides yields $\overline{A_\mathbb{C}\xi} = \overline{\lambda\xi}$. By multiplicativity of the complex conjugate, we get $\overline{A_\mathbb{C}}\overline{\xi} = \overline{\lambda}\overline{\xi}$. We know that $\overline{A_\mathbb{C}} = A_\mathbb{C}$ since $A_\mathbb{C}$ has real entries. Further, since ξ is nonzero, then $\overline{\xi}$ must also be nonzero. Let's denote these new eigenvalues as $\xi^* := \overline{\xi}$. Hence, $A_\mathbb{C}\xi^* = \overline{\lambda}\xi^*$ and hence $\overline{\lambda}$ is an eigenvalue of A. $\square$

Now, equipped with the ideas of complexification and semisimplicity, we will discuss an alternative approach to diagonalization for matrices with complex eigenvalues.

Definition 5 (Normal Form)

Let $T \in \mathcal{L}(\mathbb{R}^n)$. The real normal form of T is a block diagonal matrix N such that for each eigenvalue λ:

1. If $\mathrm{Im}(\lambda) = 0$ then N has a 1×1 block $[\lambda]$ on the diagonal.
2. Otherwise, each complex conjugate pair of eigenvalues $a \pm bi$ corresponds to a 2×2 block of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

More formally, if $T \in \mathcal{L}(\mathbb{R}^n)$ has r real eigenvalues $\lambda_1, ..., \lambda_r$ and $2k$ complex eigenvalues $\mu_1, \overline{\mu_1}, ..., \mu_k, \overline{\mu_k}$ (with each $\mu_j = a_j + b_j i$ then there exists a transformation $N \in \mathcal{L}(\mathbb{R}^n)$ similar to T of the form:

$$\begin{bmatrix} \lambda_1 & & & & & & & \\ & \ddots & & & & & & \\ & & \lambda_r & & & & & \\ & & & a_1 & -b_1 & & & \\ & & & b_1 & a_1 & & & \\ & & & & & \ddots & & \\ & & & & & & a_k & -b_k \\ & & & & & & b_k & a_k \end{bmatrix}$$