8. Eigenstuff

This week we will discuss eigenvalues, eigenvectors, and eigenspaces. I often refer to these collectively as "eigenstuff" or a more vulgar alternative. We'll start with a ton of definitions, most of which will likely be familiar to some extent from a previous linear algebra course.

Definition 1 (Eigenvector)

An eigenvector is a vector ξ for which $A\xi = \lambda\xi$. Conceptually, an eigenvector is a vector which, when a linear transformation is applied, is only scaled or possibly negated.

Definition 2 (Eigenvalue)

An eigenvalue λ is a scalar quantity associated with a given eigenvector ξ which represents the scaling factor in the equation $A\xi = \lambda\xi$.

Definition 3 (Spectrum)

The spectrum of a matrix A, denoted $\lambda(A)$, is the set of all eigenvalues.

Definition 4 (Characteristic Equation)

The characteristic equation of a matrix A, denoted $p_A(\lambda) := \det(A - I\lambda)$.

Definition 5 (Eigenspace)

If λ is an eigenvalue of A, the the subspace $\ker A - I\lambda$ is the eigenspace of λ.

Note that when calculating, you should first calculate the eigenvalues, then the eigenvectors, then the eigenspaces.
We shall now write some definitions for multiplicities.

Definition 6 (Algebraic Multiplicity)

If λ is an eigenvalue of A then it is a root of $p_A(t) = \det(A - It)$. The multiplicity of the root is the algebraic multiplicity.

Definition 7

If λ is an eigenvalue of A then nullity$(A - I\lambda)$ is called the geometric multiplicity.

Theorem 1

The geometric multiplicity is less than or equal to the algebraic multiplicity.

We shall end with two theorems that relate the eigenvalues to other fundamental matrix operations.

Theorem 2

If $\lambda(A) = \{\lambda_1, ..., \lambda_n\}$ with algebraic multiplicities $\alpha_1, ..., \alpha_n$. Then $\operatorname{tr} A = \lambda_1 + \lambda_2 + \cdots + \lambda_2$.

Theorem 3

If $\lambda(A) = \{\lambda_1, ..., \lambda_n\}$ with algebraic multiplicities $\alpha_1, ..., \alpha_n$. Then $\det(A) = \lambda_1\lambda_2 \cdots \lambda_2$.

Proof.
We know that the characteristic equation is $\det(A - I\lambda) = (\lambda_1 - \lambda)^{\alpha_1}(\lambda_2 - \lambda)^{\alpha_2} \cdots (\lambda_n - \lambda)^{\alpha_n}$. If we set $\lambda = 0$ we get $\det(A) = \lambda_1^{\alpha_1}\lambda_2^{\alpha_n} \cdots \lambda_n^{\alpha_n}$ $\square$