7. The Determinant

The determinant of a matrix is associated with permutations of elements, selecting each element such that no two elements share the same row or column.
We denote the determinant as $D(v_1, ..., v_n)$ where $D : V^n \to \mathbb{K}$ and $v_1, ..., v_n \in V$ for vector space V and scalar field $\mathbb{K}$.
The following properties hold true for determinants:

1. Linearity: $D(v_1, ..., v_i + \alpha v_j, ..., v_n) = D(v_1, ..., v_i, ..., v_n) + \alpha D(v_1, ..., v_j, ..., v_n)$.
2. Preservation under replacement: $D(v_1, ..., v_i + \alpha v_k, ..., v_k, ..., v_n) = D(v_1, ..., v_i, ..., v_k, ..., v_n)$.
3. Antisymmetry: $D(v_1, ..., v_i, ..., v_j, ..., v_k) = -D(v_1, ..., v_j, ..., v_k, ..., v_n)$.
4. Normalization: $D(e_1, ..., e_n) = 1$.

If $A = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$ then we define $\det(A) = D(v_1, ..., v_n)$.

Theorem 1

$\det(A) = \det\left(A^T\right)$

Theorem 2

$\det(AB) = \det(A)\det(B)$

We shall now formalize our notion of the determinant:

Definition 1 (Permutation)

A permutation σ of $\{x_1, ..., x_n\}$ is a rearrangement of the elements into some order. The set of all permutations on n elements is S_n.

Definition 2

Sign The sign of a permutation, denoted $\operatorname{sgn}(\sigma)$ is $+1$ if an even number of swaps is needed and -1 if an odd number of swaps is needed to return to the identity permutation.

Definition 3

Determinant The determinant of A is defined as,

$$\det(A) = D(A_1, ..., A_n) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}$$

Definition 4 (Minor)

The determinant of the $k \times k$ submatrix, taking k rows and k columns, yields the minor of order k.

Theorem 3 (Cofactor Expansion)

We can use the familiar cofactor expansion as follows:

$$D(A_1, ..., A_n) = \sum_{k=1}^{n} a_{j,k}(-1)^{j+k}\det(A_{i,k})$$

We call $C_{j,k} = (-1)^{j+k}\det(A_j, k)$ the cofactor at j, k.

Definition 5 (Cofactor Matrix)

We call $C = [C_{j,k}]$ the cofactor matrix.

Theorem 4

If C is the cofactor matrix for A then $A^{-1} = \frac{1}{\det(A)} C^\top$