5. Matrix Products and the Exponential

Today, we will do an enrichment topic which are extensions on the matrix product.
In real numbers, you may recall that we define the exponential using the Taylor series:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

We follow a similar definition to find the exponential of a matrix:

Definition 1 (Exponential of a Matrix)

The exponential of a matrix is defined as $e^A := \sum_{n=0}^{\infty} \frac{A^n}{n!} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$.

We shall now introduce a few other types of matrix products:

Definition 2 (Kronecker Product)

We define the Kronecker product $A \otimes B$ of two matrices A and B to be,

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}$$

So, if A is $m \times n$ and B is $k \times \ell$, then $A \otimes B$ is $km \times \ell n$.

Example 1

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$$

$$A \otimes B = \begin{bmatrix} 2 & 1 & -4 & -2 \\ 5 & 4 & -10 & -8 \\ 6 & 3 & -8 & -4 \\ 15 & 12 & -20 & -16 \end{bmatrix}$$

Definition 3 (Schur Product)

We define the Schur product (or "bad students' product") $A \circ B$ to be $(A \circ B)_{ij} = A_{ij}B_{ij}$. It is only defined for two matrices of the same size.

Example 2

$$A = \begin{bmatrix} 2 & 1 & 3 & 2 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 2 & 4 & 3 & 1 \\ 1 & 3 & 1 & 4 \\ 2 & 1 & 3 & 2 \end{bmatrix}$$

$$A \otimes B = \begin{bmatrix} 4 & 4 & 9 & 2 \\ 1 & 6 & 3 & 8 \\ 6 & 1 & 6 & 6 \end{bmatrix}$$

I will end this part with a question for you to attempt. What are the identities for the Kronecker and Schur products? That is, find $I_\circ$ so that $I_\circ \circ A = A \circ I_\circ = A$ and $I_\otimes$ so that $I_\otimes \otimes A = A \otimes I_\otimes = A$.