4. Fun with Isomorphisms

This week, we discuss isomorphisms of vector spaces. First, we shall discuss invertibility.

Definition 1 (Invertibility)

A map f is invertible if we can find some g such that $f \circ g = \text{id}$.

Theorem 1

A map is invertible if and only if it is bijective (i.e., surjective and injective).

Now we can discuss the concept of isomorphisms of vector spaces:

Definition 2 (Isomorphism)

Let V and W be vector spaces. Let $T : V \to U$ be a linear transformation. If T is invertible, then it is called an isomorphism.

Note that the term "isomorphism" means "same shape" in Greek. This may give you an idea of why it is relevant.

Definition 3 (Isomorphic)

We say that two vector spaces U and V are isomorphic if we can find some isomorphism $T : U \to V$ (which also implies that there exists an isomorphism $S : V \to U$). We write $U \simeq V$.

Isomorphisms are handy as they allow us to "translate" a problem in a difficult vector space U to an easier vector space V. The typical workflow would be to start with your problem in U, translate the problem to an equivalent problem in V, solve the problem in V, and then translate your answer back to U.