3. Linear Transformations

In this part, we'll put "linear" in linear algebra! But first, we must state a few definitions.

Definition 1 (Additivity)

Let V and W be vector spaces. A map $T : V \to W$ is called additive if $T(u+v) = T(u) + T(v)$ for all $u, v \in V$

Definition 2 (Homogeneity)

Let U and W be vector spaces. A map $T : V \to W$ is called homogeneous if $T(ku) = kT(u)$ for all $u \in V$ and $k \in \mathbb{K}$.

Definition 3 (Linearity)

We say a map is linear if it is homogeneous and additive.

Definition 4 (Operator)

An operator is a transformation $T : V \to \ V$ whose domain and codomain are equal.

We can write a linear transformation as a matrix!

Example 1

Consider $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that T reflects a vector over $y = x$. This is like swapping the x and y components. To write the matrix, take each standard basis element and apply the transformation. Here,

$$T(e_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and,

$$T(e_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

We put each of the results as a column in the matrix representation. That is, in general, $\big[T\big] = \big[T(e_1) \quad T(e_2) \quad \cdots \quad T(e_n)\big]$. So here, we have,

$$\big[T\big] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

We shall now describe a few definitions associated with linear transformations:

Definition 5 ($\mathcal{L}(V, W)$)

Let V and W be vector spaces. The set of linear transformations from V to W is $\mathcal{L}(V, W)$.

Definition 6 (Kernel)

Let V and W be vector spaces, and let $T \in \mathcal{L}(V, W)$. Then the kernel of T, denoted $\ker T$, is the set of vectors which get mapped to 0 under T.

Theorem 1

Let $T \in \mathcal{L}(V, W)$ for vector spaces V, W. Then $\ker T$ is a subspace of V.

Definition 7 (Image)

For any $T \in \mathcal{L}(V, W)$, the image of T, denoted $\text{im}\,(T)$, is the set of vectors that "can be reached" by T.

Definition 8 (Nullity)

The nullity of T is nullity $(T) := \dim(\ker T)$.

Definition 9 (Rank)

The rank of T is rk $(T) := \dim(\text{im}(T))$.

Theorem 2 (Rank-Nullity Theorem)

$$\dim(V) = \text{rk}(T) + \text{nullity}(T).$$

We shall end this part with a reminder on some definitions related to maps in general, which can also be applied to linear transformations.

Definition 10 (Domain and Codomain)

A map $f : A \to B$ has domain A and codomain B.

In the above definition, it is possible that f cannot actually attain all values of B. For example, consider $f(x) = x^2$. Perhaps $f : \mathbb{R} \to \mathbb{R}$, so $\text{cod}(f) = \mathbb{R}$, but $\text{im}(f) = \mathbb{R}_{\geq 0}$.

Definition 11 (Surjectivity)

If a map can attain its whole codomain then we call it surjective. That is, if $\text{im}(f) = \text{cod}(f)$.

Definition 12 (Injectivity)

A map which satisfies $f(x) = f(y) \implies x = y$ for all $x, y \in \text{dom}(f)$ is called injective.

Definition 13 (Bijectivity)

A map which is injective and surjective is called bijective.