2. Basis and Dimension

Today we will start with something abstract.

Definition 1 (Linear Combination)

Suppose that we have n "symbols": $x_1, x_2, ..., x_n$. We say that $\sum_{i=0}^{n} \alpha_i x_i$ represents some linear combination.

Definition 2 (Span)

The set of all vectors which can be reached using a linear combination of a set of vectors is called the span of those vectors. That is,

$$\text{span} v_1, v_2, ..., v_n = \{\alpha_1 v_1 + ... + \alpha_n v_n : \alpha_i \in \mathbb{K} \quad \forall i \in \{1, ..., n\}\}$$

Definition 3 (Generating Set)

We say that a set $\{v_1, ..., v_n\}$ is generating if its span is equal to the entire vector space.

In your first course in linear algebra, we said that a set was linearly independent if v_i could not be written as a linear combination of the other vectors. In this course, we use a slightly different definition:

Definition 4 (Linear Independence)

We say that $\{v_1, ..., v_n\}$ is linearly independent if and only if $\sum_{i=1}^{n} \alpha_i x_i = 0 \iff \alpha_1 = \cdots = \alpha_n = 0$. (That is, the only solution to this system of equations is the trivial solution.)

Recall the following definition of a basis:

Definition 5 (Basis)

A set of vectors is called a basis for a vector space V if and only if it is linearly independent and generating.

Definition 6 (Dimension)

The dimension is the number of vectors in any basis (which will always be the same, regardless of basis).

A good way to think about dimension:

- A set of vectors will always start out linearly independent (consider $\{v\}$, a singleton), and as we add vectors to the set it will become linearly dependent at some point, at latest after some "magic number" of vectors have been added.
- A set of vectors will start out as a span "smaller" than the vector space itself. As you add vectors, the "size" of the vector space must plateau at a minimum of some "magic number" of vectors.

This "magic number" is the dimension.
Note: It is possible to have ∞-dimensional vector spaces! Consider $\mathbb{R}^{\mathbb{R}}$