11. Inner Product Spaces

This time, we will talk about an "enhanced" type of vector space – the inner product space.

Definition 1 (Inner Product Space)

An inner product space $(V, +, *, \langle \cdot, \cdot \rangle)$ is a vector space paired with a product called an inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{K}$ such that:

1. $\langle \cdot, \cdot \rangle$ is conjugally symmetric, i.e., $\langle u, v \rangle = \overline{\langle v, u \rangle}$.
2. $\langle \cdot, \cdot \rangle$ is linear in the first slot, i.e., $\langle ku + v, w \rangle = k\langle u, w \rangle + \langle v, w \rangle$.
3. $\langle \cdot, \cdot \rangle$ is positive definite, i.e., $\langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0$ if and only if $v = 0$.

From this, we can define three quantities:

Definition 2 (Norm)

The norm of a vector $v \in V$ is $|| \cdot || : V \to \mathbb{K}$ such that $||v|| = \sqrt{\langle v, v \rangle}$.

Definition 3 (Angle)

We define an abstract version of the angle between two vectors using $\langle u, v \rangle = ||u|| ||v|| \cos \theta$.

Definition 4 (Metric)

We define the metric of two vectors as the function $m : V^2 \to \mathbb{K}$ such that $m(u, v) = ||u - v|| = \sqrt{\langle u - v, u - v \rangle}$.

Quick note: We have conjugate symmetry of $\langle \cdot, \cdot \rangle$ in general. For real inner product spaces, we have full symmetric.
If $z = a + ib$ then $\overline{z} = a - ib$. So to take the conjugate of a complex number, we just swap the sign of the imaginary part.
Two key inequalities hold for norms:

1. Cauchy-Schwarz Inequality: $|\langle u, v \rangle|$
2. Triangle Inequality: $||u + v|| \leq ||u|| + ||v||$

As practice, we will prove the reverse triangle inequality:

Theorem 1

For any vectors v, u in some inner product space V, we have $|||v|| - ||u||| \leq ||v - u||$.

Proof. Let u, v be vectors in an inner product space V. Then we know the triangle inequality holds so $||u|| + ||v - u|| \geq ||u + v - u|| = ||v||$ and $||v|| + ||u - v|| \geq ||u||$ without loss of generality. Rearranging yields $||v - u|| \geq ||v|| - ||u||$ and $||u - v|| \geq ||u|| - ||v||$. Notice that $||u - v|| = ||v - u|| = m(u, v)$ and so let's take the absolute value of both sides to get $||v - u|| \geq |||v|| - ||u|||$. □

Some common inner products are as follows:

- On $\mathbb{R}^n$: $\langle v, u \rangle = \sum_i v_i u_i$
- On $\mathbb{C}^n$: $\langle v, u \rangle \sum_i v_i \overline{u_i}$
- On $\mathbb{R}^{\mathbb{R}}$: $\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) g(x) \, \mathrm{d}x$