1. Proof and Vector Space Basics

Welcome to Advanced Linear Algebra! First, we will discuss basic proof techniques. The following are some useful proof techniques you may have learned about in a previous course:

1. Direct proof
2. Proof by contraposition (e.g., when it rains, the sidewalk gets wet. The sidewalk is currently dry. What can we conclude?
3. Proof by contradiction. (N.B., this can take many forms!)
4. Weak induction (e.g., how can we prove that given a setup of dominoes, all dominoes will fall?)

It is often fruitful to write out your assumptions and what you want to show. This may help give you a "stroke of genius".
Tips for when you're stuck:

- Go for a nice walk (at least 1 hour, preferrably in nature).
- Call your mom/dad/sibling/friend.
- Go for coffee.

Remember that undergraduate students often make movement on open problems! Their fresh perspective allows them to see things that lifelong mathematicians may not consider.
We shall now shift over to vector spaces.

Definition 1 (Vector Space)

A vector space $(V, +. *)$ is a set V along with two operations $+$ and $*$ such that the following axioms hold:

1. V is closed under addition, i.e., if $u, v \in V$ then $u + v \in V$.
2. V is closed under scalar multiplication, i.e., if $u \in V$ and $k \in \mathbb{K}$ then $ku \in V$.
3. V is commutative under addition, i.e., $u + v = v + u \quad \forall u, v \in V$.
4. V is associative under addition, i.e., $(u + v) + w = u + (v + w) \quad \forall u, v, w \in V$.
5. V has an additive identity, i.e., there is a $z \in V$ such that for all $v \in V$, $z + v = v$.
6. V has additive inverses, i.e., for every $v \in V$ there is a $w \in V$ such that $v + w = z$.
7. V has a scalar multiplicative identity, i.e., $1v = v$.
8. V is distributive for scalar multiplication over addition, i.e., $a(u + v) = au + av$ and $(a + b)u = au + bu$ for $u, v \in V$ and $a, b \in \mathbb{K}$.

What is that $\mathbb{K}$? Well, $\mathbb{K}$ is a field, which is essentially a set which permits addition, subtraction, multiplication, and division.

Definition 2 (A^B)

If A and B are sets, then A^B is the set of all functions from B to A, i.e., $A^B := \{f | f : B \to A\}$.

Definition 3 (Subspace)

A subspace is a subset of a vector space which is, itself, a vector space.

Instead of proving some set W is a vector space, just find some larger vector space V such that $W \subseteq V$, then use the 2-step subspace test.

Theorem 1 (2-Step Subspace Test)

Let V be a vector space and let $V \supseteq W \neq \{\}$. Then W is a subspace if and only if W is closed under both addition and scalar multiplication.